Polynomial Approximations for the Electric Polarizabilities of Some Small Apertures

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Abstract —**Polynomial expressions are given for the electric polarizabilities of some small apertures of various shapes, as functions of the aperture width to length ratios. The shapes considered are rectangle, diamond, rounded end slot, and ellipse, of which only the last is known to have an exact solution. Although the polynomial expressions are not exact, all embody features which would exist in exact solutions if they could be found. Values calculated from the polynomials compare well with previously published data, indicating accuracy sufficient for many purposes.**

I. Introduction

In several branches of electromagnetic engineering there is a need to determine the polarizability of small apertures of various shapes. This paper is concerned with the electric polarizabilities of small apertures of the shapes shown in Fig. 1, of which only the ellipse is known to have an exact solution. All of the shapes in Fig. 1 are characterized by a maximum length L and a maximum width W, and the width to length ratio or aspect ratio W/L will in all cases be designated α.

A common convention is for magnetic polarizabilities to be positive quantities and for electric polarizabilities to be negative quantities. For simplicity in this paper dealing only with electric polarizabilities, positive quantities will be used throughout.

II. Rectangle

In a recent paper, Arvas and Harrington [1] have given numerical values for the electric polarizabilities of rectangular apertures of various aspect ratios as an example of their technique for computing the electric polarizabilities of apertures as the dual of the magnetic polarizabilities of conducting disks.

It is of interest to compare their values with those calculated earlier [2], [3] using a variational modal technique. The electric polarizability of a rectangular aperture of side lengths L and W, as in Fig. 1(a), may be expressed as

$$Pe = R_E L^3 \tag{1}$$

in which the coefficient R_E is a function of the aspect ratio

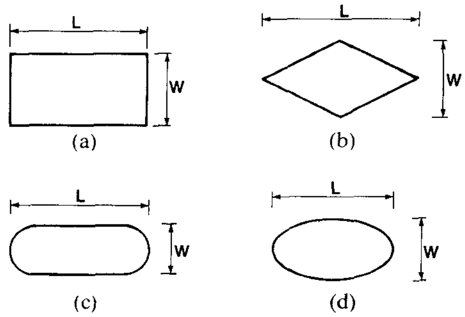

Fig. 1. Aperture Shapes. (a) Rectangle. (b) Diamond. (c) Rounded end slot. (d) Ellipse.

W/L, i.e.,

$$R_E = f\left(\frac{W}{L}\right). \tag{2}$$

In Table I, the numerical values for R_E from Arvas and Harrington [1] are compared with those from the earlier work [2], [3]. Also shown in Table I are the values calculated from a simple polynomial to be discussed below, and Cohn's experimental values [4].

It will be noted that there is good agreement between the 1983 and 1971 solutions, and with the experimental results from [4].

The polarizability of a square ($W = L$) is of particular interest, not only because of the symmetry of the problem but also because that value determines the slope of a function in addition to its magnitude as will now be shown.

The electric polarizability of a rectangular aperture is independent of the choice of which side is L and which is W as it is associated only with the normal electric field at the aperture. (In the case of the magnetic polarizability, the sides have to be related to the direction of the tangential magnetic field.) Thus, from (1) and (2)

$$f\left(\frac{W}{L}\right)L^3 = f\left(\frac{L}{W}\right)W^3 = \left(\frac{W}{L}\right)^3 f\left(\frac{L}{W}\right)L^3.$$

Thus, if the aspect ratio W/L is designated α

$$f(\alpha) = \alpha^3 f\left(\frac{1}{\alpha}\right) \tag{3}$$

and if an analytical solution to this problem was ever found, it would satisfy (3).

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TABLE I
ELECTRIC POLARIZABILITY COEFFICIENT OF A RECTANGULAR APERTURE AS A FUNCTION OF ASPECT RATIO

α	Arvas & Harrington 1983	McDonald 1971	Polynomial	Cohn Experiment 1952
1.0	0.1116	0.1126	0.1126	0.1137
0.9	-	0.0960	0.0960	-
0.8	-	0.0799	0.0800	-
0.75	0.0717	-	0.0722	0.0731
0.7	-	0.0645	0.0647	-
0.6	-	0.0501	0.0502	-
0.5	0.0364	0.0368	0.0369	0.0370
0.4	-	0.0249	0.0250	-
0.3	0.01468	0.0148	0.0149	0.0147
0.2	0.0069	0.00695	0.00701	0.0070
0.1	0.00184	0.00183	0.00186	0.0019

Taking the derivative of (3) with respect to α gives

$$f'(\alpha)=3\alpha^2 f\left(\frac{1}{\alpha}\right)-\alpha f'\left(\frac{1}{\alpha}\right)$$

which, for $\alpha=1$ (a square aperture), reduces to

$$f'(1)=\frac{3}{2}f(1). \tag{4}$$

Therefore, a numerical value for the polarizability coefficient for a square aperture, together with the knowledge [4], [5] that as

$$\alpha\to 0 \qquad f(\alpha)\to\frac{\pi}{16}\alpha^2 \tag{5}$$

gives a considerable amount of information about $f(\alpha)$. For example, if $f(\alpha)$ is approximated by the polynomial

$$f(\alpha)=a+b\alpha+c\alpha^2+d\alpha^3+e\alpha^4$$

for α in the range 0 to 1, from (5) $a=0$, $b=0$, and $c=\pi/16$. Then d and e can be determined from (4) with $f(1)$ taken as 0.1126. (The Arvas and Harrington value of 0.1116 may increase slightly if more interior nodes are used, as indicated in their paper.) The resulting polynominal expression for $f(\alpha)$, which is the polarizability coefficient R_E, can then be expressed as

$$f(\alpha)=\frac{\pi}{16}\alpha^2\{1.0-0.5663\alpha+0.1398\alpha^2\}. \tag{6}$$

Values for $f(\alpha)$ calculated from (6) are given in Table I and show good agreement with the numerical solutions and experimental results. Accordingly, the polynomial expression (6) should be sufficiently accurate for many purposes.

Note that as the polynomial expression (6) does not satisfy (3), it is not valid for all α. (It is simply the lowest order polynomial approximation for $0\le\alpha\le 1$ for which all coefficients can be determined from the values of the function for $\alpha\to 0$ and $\alpha=1$.) However, that presents no limit to its utility, as an aspect ratio of greater than 1 can always be converted to less than 1 by (3), or more simply L is chosen to be the longer side so that W/L is always less than 1.

Because of the apparent success of the fourth-power polynomial approach for the rectangular aperture, the question arises whether it could give useful results for other shapes. In particular, there would be interest in the diamond and the rounded end slot of Fig. 1(b) and (c). However, for these shapes, the coefficients in the fourth-power polynomial cannot be determined so directly and some intuitive reasoning is required to find some of the coefficients.

III. DIAMOND

For the diamond shape in Fig. 1(b), the electric polarizability can be expressed as

$$Pe=g(\alpha)L^3$$

in which the polarizability coefficient $g(\alpha)$ is a function of the aspect ratio W/L.

For the diamond, the choice of L or W for the reference direction is arbitrary, as it is for the rectangle, leading to

$$g'(1)=\frac{3}{2}g(1).$$

Thus, if the electric polarizability of a square is considered to be known to good accuracy from Section II, two equations are available for the determination of the coefficients of the fourth-power polynomial. However, whereas for the rectangular aperture the behavior was known for $\alpha\to 0$, for the diamond shape some intuitive reasoning is necessary to ascertain the small α behavior.

For a rectangular aperture, as the ratio of width W to length L goes to zero

$$Pe\to\frac{\pi}{16}\left(\frac{W}{L}\right)^2L^3 = \frac{\pi}{16}W^2L$$

which may be interpreted as a polarizability of $(\pi/16)W^2$ per unit length [5].

This suggests that if the width ω of a long narrow aperture varies very slowly along the length, then the polarizability could be obtained by integrating $(\pi/16)\omega^2$ along the length of the aperture.

This postulate is supported by the fact that if it is used to calculate the electric polarizability of a very long narrow ellipse (as in Fig. 1(d) but with $W\ll L$) the result is

$$\frac{\pi}{24}W^2L$$

which agrees with the exact solution from [6] for an ellipse of eccentricity approaching unity.

The application of this reasoning to the diamond shape gives

$$\frac{W}{L}\to 0 \qquad Pe\to\frac{\pi}{48}W^2L$$

i.e.,

$$\alpha\to 0 \qquad g(\alpha)\to\frac{\pi}{48}\alpha^2.$$

If, for α in the range $0 \le \alpha \le 1$, $g(\alpha)$ is approximated by

$$g(\alpha) = a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4$$

then

$$a = 0, b = 0, \text{ and } c = \frac{\pi}{48}.$$

Also, using the results for a square from Section II for a diamond with $W = L$, the polarizability is

$$0.1126\left(\frac{L}{\sqrt{2}}\right)^3$$

leading to $g(1) = 0.0398$ and $g'(1) = 0.0597$.

Then the resulting polynomial can be put in the form

$$g(\alpha) = \frac{\pi}{48}\alpha^2\{1.0 - 0.4794\alpha + 0.0876\,\alpha^2\}. \quad (7)$$

Possibly the best check on this expression from the published literature is in [1, fig. 5]. In that figure, the "normalized electric polarizability" T_{av}, defined as the polarizability divided by $(\text{area})^{3/2}$, is shown for both rectangular and diamond shapes. For an aspect ratio of 1, T_{av} for a rectangle and a diamond are the same, as in that case both are squares. For smaller aspect ratios, T_{av} for a diamond is slightly less than for a rectangle, with the difference increasing as the aspect ratio is decreased until, at $\alpha = 0.1$, the difference is approximately 5 percent. The ratio between T_{av} for a diamond and T_{av} for a rectangle calculated from the polynomial approach taken in this paper would range between 1.0 for $\alpha = 1.00$ and $2\sqrt{2}/3$ ($= 0.943$) for $\alpha \to 0$, with intermediate values of 0.95 at $\alpha = 0.1$ and 0.98 at $\alpha = 0.5$.

Also, [1, fig. 4] contains a plot, generated from six computed data points, of the polarizability coefficient against aspect ratio for diamond-shaped apertures. If values of $g(\alpha)$ from (7) for the same six aspect ratios are also plotted on that figure, they fall within the small circles marking the data points.

IV. Rounded End Slot

If, for the rounded end slot shown in Fig. 1(c), the electric polarizability is expressed as

$$Pe = h(\alpha)L^3$$

then $h(1) = 1/12$ as that is the known result for a circle [6]. Also, as $\alpha \to 0$, $h(\alpha)$ for a rounded end slot will approach the same value as for a rectangle. (Consider a rounded end slot of fixed width W and length L. As L is increased, the polarizability will go to $(\pi/16)W^2L$ plus a correction term. As $L \to \infty$, i.e., $\alpha \to 0$, the $(\pi/16)W^2L$ term will dominate.) Therefore, as

$$\alpha \to 0 \qquad h(\alpha) \to \frac{\pi}{16}\alpha^2.$$

What is missing in this case is a direct way of obtaining $h'(1)$ as there is no equivalent of (3) ($\alpha > 1$ has no meaning).

Consider the effect on the polarizability of a circular aperture of radius R if the radius is reduced very slightly

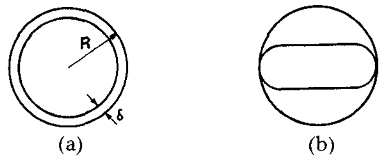

Fig. 2. (a) Circle within a circle. (b) Rounded end slot within a circle.

TABLE II
ELECTRIC POLARIZABILITY COEFFICIENT OF A ROUNDED END SLOT AS A FUNCTION OF ASPECT RATIO

α	Polynomial	Cohn Experiment
1.0	0.0833	0.0833
0.75	0.0589	0.0585
0.5	0.0326	0.0325
0.3	0.0139	0.0143
0.2	0.0067	0.0070
0.15	0.0039	0.0041
0.1	0.0018	0.0019

by δ, as in Fig. 2(a)

$$Pe = \frac{2}{3}(R - \delta)^3$$

$$\approx \frac{2}{3}R^3 - 2R^2\delta.$$

Thus, for incrementally small values of δ, the change in Pe is proportional to δ and is R/π times the change in area. Also, it is known that the electric polarizability is not orientation-dependent. This suggests that if δ is not uniform around the boundary, provided it is very small and its variation is smooth, then the change in polarizability may be obtained by integrating δ around the circumference, i.e., from R/π times the change in area. This postulate is supported by the fact that if it is applied to an elliptical aperture of very small eccentricity, by considering the ellipse to be a slight deformation of a circle, the result for the polarizability is

$$\frac{1}{24}(3W - L)L^2$$

which agrees with the exact solution from [6] for an ellipse of eccentricity approaching zero.

Then $h'(1)$ can be obtained by considering the limit as $\alpha \to 1$ of the rounded end slot within a circle as shown in Fig. 2(b). The result is

$$h'(1) = \frac{1}{4} - \frac{1}{2\pi}.$$

The solution then proceeds as for the other aperture shapes to give

$$h(\alpha) = \frac{\pi}{16}\alpha^2\{1.0 - 0.7650\alpha + 0.1894\alpha^2\}. \quad (8)$$

Values for $h(\alpha)$ calculated from (8) are compared in Table II with Cohn's experimental results [4].

Note that if the reasoning used to obtain $h'(1)$ was incorrect, errors in $h(\alpha)$ could be expected for the larger

TABLE III
ELECTRIC POLARIZABILITY COEFFICIENT OF AN ELLIPSE AS A FUNCTION OF ASPECT RATIO

α	Polynomial	Exact Value	Error
1.0	0.0833	0.0833	-
0.9	0.0709	0.0710	0.1%
0.8	0.0588	0.0591	0.5%
0.7	0.0472	0.0477	1.0%
0.6	0.0363	0.0369	1.6%
0.5	0.0264	0.0270	2.2%
0.4	0.0177	0.0182	2.7%
0.3	0.0104	0.0107	2.8%
0.2	0.00482	0.00498	3.2%
0.1	0.00126	0.00129	2.3%

The largest absolute error occurs for α between 0.5 and 0.6, and the largest percentage error occurs for α of approximately 0.2.

values of α. However, the values of $h(\alpha)$ for $\alpha = 0.5$ and $\alpha = 0.75$ are both within 1 percent of the experimental results.

V. ELLIPSE

Finally, there may be some interest in a fourth-power polynomial expression for the electric polarizability of an elliptical aperture, either as a simpler alternative to the exact solution [6] containing an elliptic integral, or to ascertain for this case how close the polynomial approach is to the exact solution (the ellipse is the only shape for which this test can be applied).

For an ellipse, the polynomial expression for the polarizability coefficient, obtained by the methods outlined above, is

$$i(\alpha) = \frac{\pi}{24}\alpha^2\{1.0 - 0.4085\alpha + 0.0451\alpha^2\}. \qquad (9)$$

From the comments made in Sections III and IV, it is known that $i(\alpha)$ has the correct behavior as $\alpha \to 0$, and has the correct magnitude and slope at $\alpha = 1$.

In Table III, values for the polarizability coefficient calculated from the polynomial (9) are compared with those from the exact solution.

VI. CONCLUSIONS

Polynomial approximations for the electric polarizability coefficients of some small apertures are presented in (6)–(9). When multiplied by L^3, those expressions give the electric polarizabilities of the respective apertures.

It has been assumed that the apertures are small in wavelength and that the wall is of infinitesimal thickness. If either of those conditions is not satisfied, correction terms will be necessary [7].

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